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On Bivariate Birkhoff Interpolation

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1. INTRODUCTION

In this paper we will give criteria for unique solvability of multivariate Birkhoff interpolation problems. In the bivariate case we mean by Birkhoff interpolation the following problem: Let $\mathscr{C}^p(G)$, with $p \in \mathbb{N}_0$ and $G \subset \mathbb{R}^2$ and compact, be the space of all real valued functions continuously differentiable of total order p on G, let $\prod_s$, $s \in \mathbb{N}$, be the space of all polynomials of degree $\leqslant s$ in one real variable, and let $D^{i,j}_{x,y}$ for $(x, y) \in G$, be the functional $D^{i,j}_{x,y}(f) = f^{(i,j)}(x, y)(f \in \mathscr{C}^{i+j}(Q))$. Let now m, M, $r \in \mathbb{N}_0$, $Z \subset \{1,..., m\} \times \{0,..., M\}$ with $\operatorname{card}(Z) = r+1$, and consider the base set of knots (cf. Definition 3.1)

$$K = \{(x_i, y_{i,k;j}) \mid (i, k) \in Z,\ 1 \leqslant j \leqslant a_{i,k}\} \subset G$$

with $a_{i,k} \in \mathbb{N}_0$ and the sets

$$L_{i,k} \subset \{1,..., a_{i,k}\} \times \{0,..., M-k\}$$

for $(i, k) \in Z$. Moreover, let $(i_0, k_0),..., (i_r, k_r)$ be an ordering of Z with $\operatorname{card}(L_{i_s,k_s}) \geqslant \operatorname{card}(L_{i_t,k_t})$ for $0 \leqslant s < t \leqslant r$. We investigate whether the problem

$$\begin{aligned}&\Big(\mathscr{C}^M(G), \sum_{s=0}^{r} \Pi_s \otimes \Pi_{\operatorname{card}(L_{i_s,k_s})}; D^{k,l}_{x_i,y_{i,k;j}}:\\ &(i, k) \in Z, (j, l) \in L_{i,k}, (x_i, y_{i,k;j}) \in K\Big)\end{aligned} \tag{1.1}$$

is uniquely solvable, i.e., whether for every function $f \in \mathscr{C}^M(G)$ there exists exactly one $P \in \sum_{s=0}^{r} \prod_s \otimes \prod_{\operatorname{card}(L_{i_s},k_s)}$ with $D^{k,l}_{x_i,y_{i,k;j}}(P) = D^{k,l}_{x_i,y_{i,k;j}}(f)$ for all $(i, k) \in Z$, $(j, l) \in L_{i,k}$ and $(x_i, y_{i,k;j}) \in K$ (For the theory of interpolation refer to Davis [4]).

To this end we introduce a method which interpolates with tensor-

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functionals. It is similiar to the method used by Haussmann [12], but it yields an interpolation space independent of the dual functions of the functionals applied and it therefore gives sufficient conditions for unique solvability of the problem (1.1). We define and use two-dimensional incidence matrices corresponding to a problem (1.1). These matrices must be investigated for **(conditional) regularity** (cf. [14]).

2. The Interpolation Method

For $s=1,\ldots,M$, $M\in\mathbb{N}$, let F_s, G_s be finite dimensional vector spaces as well as spaces F, G with

$$\begin{aligned} &F_1\subset\cdots\subset F_M\subset F, && \dim F_s=m_s, && m_1<\cdots<m_M,\\ &G\supset G_1\supset\cdots\supset G_M, && \dim G_s=n_s, && n_1>\cdots>n_M. \end{aligned} \tag{2.1}$$

Further, let $\varphi_1\ldots,\varphi_{m_M}\in F^*$ and $\psi_{i,j}\in G^*$, $1\leqslant i\leqslant m_M$, $1\leqslant j\leqslant n_{r(i)}$ (F^*, G^* denote the dual spaces of F, G and r is the map

$$r\colon \{1,\ldots,m_M\}\ni i\to\min\{s\mid i\leqslant m_s\}\in\{1,\ldots,M\}),$$

be functionals so that the interpolation problems

$$U_s=(F,F_s;\varphi_1,\ldots,\varphi_{m_s}),\qquad 1\leqslant s\leqslant M, \tag{2.2}$$

and

$$V_i=(G,G_{r(i)};\psi_{i,1},\ldots,\psi_{i,n_{r(i)}}),\qquad 1\leqslant i\leqslant m_M, \tag{2.3}$$

are uniquely solvable.

Theorem 2.1. *The interpolation problem*

$$W=\left(F\otimes G,\sum_{s=1}^{M}F_s\otimes G_s;\varphi_i\otimes\psi_{i,j}\colon 1\leqslant i\leqslant m_M,1\leqslant j\leqslant n_{r(i)}\right) \tag{2.4}$$

is uniquely solvable.

Proof. With $m_0:=0$ the dimension of $H:=\sum_{s=1}^{M}F_s\otimes G_s$ can be determined by

$$\dim H=\sum_{s=1}^{M}(m_s-m_{s-1})\cdot n_s. \tag{2.5}$$

Since the interpolation problems U_s, $1\leqslant s\leqslant M$, and V_i, $1\leqslant i\leqslant m_M$, are uniquely solvable, there exists for every $s\in\{1,\ldots,M\}$ the dual basis

$\{f_{s,i} \mid 1 \leqslant i \leqslant m_s\}$ of $\{\varphi_{i|F_s} \mid 1 \leqslant i \leqslant m_s\}$ with reference to F_s, and for every $i \in \{1,..., m_M\}$ the dual basis $\{g_{i,j} \mid 1 \leqslant j \leqslant n_{r(i)}\}$ of $\{\psi_{i,1|G_{r(i)}},..., \psi_{i,n_{r(i)}|G_{r(i)}}\}$ with reference to $G_{r(i)}$. For $k \in \{1,..., m_M\}$, $l \in \{1,..., n_{r(k)}\}$, we define

$$\begin{aligned} a_{k,l}^{(0)} &:= f_{r(k),k} \otimes g_{k,l} \\ a_{k,l}^{(t)} &:= - \sum_{i=m_{r(k)+t-1}+1}^{m_{r(k)+t}} \sum_{j=1}^{n_{r(k)+t}} \sum_{q=0}^{t-1} (\varphi_i \otimes \psi_{i,j})(a_{k,l}^{(q)}) \\ &\quad \cdot f_{r(k)+t,i} \otimes g_{i,j}, \qquad 1 \leqslant t \leqslant M - r(k), \end{aligned} \tag{2.6}$$

and

$$h_{k,l} := \sum_{t=0}^{M-r(k)} a_{k,l}^{(t)}. \tag{2.7}$$

Then for every $t \in \{0,..., M - r(k)\}$,

$$a_{k,l}^{(t)} \in F_{r(k)+t} \otimes G_{r(k)+t} \tag{2.8}$$

is valid. In particular $h_{k,l} \in H$ for all $1 \leqslant k \leqslant m_M$, $1 \leqslant l \leqslant n_{r(k)}$. For $1 \leqslant v \leqslant m_{r(k)}$ and $1 \leqslant w \leqslant n_{r(v)}$, it follows directly from (2.8) that

$$(\varphi_v \otimes \psi_{v,w})(h_{k,l}) = \delta_{v,k} \cdot \delta_{w,l}. \tag{2.9}$$

For $v > m_{r(k)}$, $1 \leqslant w \leqslant n_{r(v)}$, the following equations are valid:

$$\begin{aligned} (\varphi_v \otimes \psi_{v,w})(h_{k,l}) &= \sum_{t=0}^{r(v)-r(k)} (\varphi_v \otimes \psi_{v,w})(a_{k,l}^{(t)}) \\ &= \sum_{t=0}^{r(v)-r(k)-1} (\varphi_v \otimes \psi_{v,w})(a_{k,l}^{(t)}) + (\varphi_v \otimes \psi_{v,w})(a_{k,l}^{(r(v)-r(k))}) \\ &= \sum_{t=0}^{r(v)-r(k)-1} (\varphi_v \otimes \psi_{v,w})(a_{k,l}^{(t)}) - \sum_{i=m_{r(v)-1}+1}^{m_{r(v)}} \sum_{j=1}^{n_{r(v)}} \\ &= \sum_{t=0}^{r(v)-r(k)-1} (\varphi_i \otimes \psi_{i,j})(a_{k,l}^{(t)})\, \varphi_v(f_{r(v),i})\, \psi_{v,w}(g_{i,j}) \\ &= \sum_{t=0}^{r(v)-r(k)-1} (\varphi_v \otimes \psi_{v,w})(a_{k,l}^{(t)}) - \sum_{t=0}^{r(v)-r(k)-1} (\varphi_v \otimes \psi_{v,w})(a_{k,l}^{(t)}). \end{aligned}$$

Therefore

$$(\varphi_i \otimes \psi_{i,j})(h_{k,l}) = \delta_{i,k} \cdot \delta_{j,l}$$

holds for $1 \leqslant i, k \leqslant m_M$, $1 \leqslant j \leqslant n_{r(i)}$, $1 \leqslant l \leqslant n_{r(k)}$; i.e., the functionals $\varphi_i \otimes \psi_{i,j}$ $1 \leqslant i \leqslant m_M$, $1 \leqslant j \leqslant n_{r(i)}$ are linearly independent in H^*. With (2.5) it follows that these functionals form a base of H^*. ▮

To derive an explicit formula of the interpolation projector

$$R\colon F\otimes G\ni h\to \sum_{i=1}^{m_M}\sum_{j=1}^{n_{r(i)}}(\varphi_i\otimes\psi_{i,j})(h)\,h_{i,j} \in \sum_{s=1}^{M} F_s\otimes G_s, \tag{2.10}$$

we define the following operators:

$$P_{s,i}\colon F\ni f\to \varphi_i(f)\,f_{s,i}\in \operatorname{span}\{f_{s,i}\},\qquad 1\leqslant s\leqslant M,\ 1\leqslant i\leqslant m_s, \tag{2.11}$$

$$Q_i\colon G\ni g\to \sum_{j=1}^{n_{r(i)}}\psi_{i,j}(g)\,g_{i,j}\in G_{r(i)},\qquad 1\leqslant i\leqslant m_M, \tag{2.12}$$

$$R_s\colon F\otimes G\ni h\to \sum_{i=1}^{m_s}\sum_{j=1}^{n_{r(i)}}(\varphi_i\otimes\psi_{i,j})(h)\,h_{s;i,j}\in \sum_{t=1}^{s} F_t\otimes G_t,\quad 1\leqslant s\leqslant M. \tag{2.13}$$

The cardinal functions $h_{s;i,j}$ can be derived from (2.7) and (2.6):

$$h_{s;i,j}=\sum_{t=0}^{s-r(i)} a_{i,j}^{(t)}. \tag{2.14}$$

Then for $2\leqslant s\leqslant M$, $1\leqslant i\leqslant m_{s-1}$, and $1\leqslant j\leqslant n_{r(i)}$, it holds that

$$\begin{aligned} h_{s;i,j}&=h_{s-1;i,j}+a_{i,j}^{(s-r(i))}\\ &=h_{s-1,i,j}-\sum_{k=m_{s-1}+1}^{m_s}\sum_{l=1}^{n_s}(\varphi_k\otimes\psi_{k,l})(h_{s-1;i,j})\,f_{s,k}\otimes g_{k,l}, \end{aligned}$$

and therefore

$$\begin{aligned} R_s(h)={}&R_{s-1}(h)+\sum_{i=m_{s-1}+1}^{m_s}\sum_{j=1}^{n_s}(\varphi_i\otimes\psi_{i,j})(h)\,h_{s;i,j}\\ &-\sum_{i=1}^{m_{s-1}}\sum_{j=1}^{n_{r(i)}}\sum_{k=m_{s-1}+1}^{m_s}\sum_{l=1}^{n_s}(\varphi_i\otimes\psi_{i,j})(h)\\ &\times(\varphi_k\otimes\psi_{k,l})(h_{s-1;i,j})\,f_{s,k}\otimes g_{k,l}\\ ={}&R_{s-1}(h)+\sum_{i=m_{s-1}+1}^{m_s}\sum_{j=1}^{n_s}(\varphi_i\otimes\psi_{i,j})(h)\,f_{s,i}\otimes g_{i,j}\\ &-\sum_{i=m_{s-1}+1}^{m_s}\sum_{j=1}^{n_s}\sum_{k=1}^{m_{s-1}}\sum_{l=1}^{n_{r(k)}}(\varphi_k\otimes\psi_{k,l})(h)\\ &\times(\varphi_i\otimes\psi_{i,j})(h_{s-1;k,l})\,f_{s,i}\otimes g_{i,j} \end{aligned}$$

$$= = R_{s-1}(h) + \sum_{i=m_{s-1}+1}^{m_s} \sum_{j=1}^{n_s} (\varphi_i \otimes \psi_{i,j})(h) f_{s,i} \otimes g_{i,j}$$

$$- \sum_{i=m_{s-1}+1}^{m_s} \sum_{j=1}^{n_s} (\varphi_i \otimes \psi_{i,j}) \left(\sum_{k=1}^{m_{s-1}} \sum_{l=1}^{n_{r(k)}} \right.$$

$$\left. \times (\varphi_k \otimes \psi_{k,l})(h)\, h_{s-1;k,l} f_{s,i} \otimes g_{i,j} \right.$$

$$= R_{s-1}(h) + \sum_{i=m_{s-1}+1}^{m_s} (P_{s,i} \otimes Q_i)(h)$$

$$- \sum_{i=m_{s-1}+1}^{m_s} (P_{s,i} \otimes Q_i)(R_{s-1}(h).$$

From this results

Lemma 2.2. *For every* $s \in \{1,\dots, M\}$,

$$R_s = R_{s-1} + \sum_{i=m_{s-1}+1}^{m_s} (P_{s,i} \otimes Q_i) \cdot (Id_{F \otimes G} - R_{s-1}). \tag{2.15}$$

Now we can verify an explicit representation formula:

Theorem 2.3. *The interpolation operator* R *of the interpolation problem* (2.4) *has the following explicit form*:

$$R = \sum_{i=1}^{m_M} P_{r(i)} \otimes Q_i + \sum_{s=1}^{M-1} (-1)^s \sum_{k_0=s+1}^{M} \sum_{k_1=s}^{k_0-1} \cdots \sum_{k_s=1}^{k_{s-1}-1}$$

$$\times \sum_{i_0=m_{k_0-1}+1}^{m_{k_0}} \cdots \sum_{i_s=m_{k_s-1}+1}^{k_{k_s}} P_{k_0,i_0} \cdot \cdots \cdot P_{k_s,i_s} \otimes Q_{i_0} \cdot \cdots \cdot Q_{i_s}. \tag{2.16}$$

(*By* $P_1 \cdot P_2$, *we mean the ordinary composition of the projections* P_1 *and* P_2.)

Proof. By induction we verify the corresponding formula for the operators R_t, $1 \leqslant t \leqslant M$. For $t = 1$,

$$R_1 = \sum_{i=1}^{m_1} P_{1,i} \otimes Q_i$$

is valid. For $1 \leqslant t \leqslant M-1$, it follows with Lemma 2.2,

$$R_{t+1} = R_t + \sum_{i=m_t+1}^{m_{t+1}} (P_{t+1,i} \otimes Q_i) \cdot (Id_{F \otimes G} - R_t)$$

$$= R_t + \sum_{i=m_t+1}^{m_{t+1}} P_{t+1,i} \otimes Q_i - \sum_{i=m_t+1}^{m_{t+1}} (P_{t+1,i} \otimes Q_i) \cdot R_t$$

$$= \sum_{i=1}^{m_t} P_{r(i),i} \otimes Q_i + \sum_{i=m_t+1}^{m_{t+1}} P_{t+1,i} \otimes Q_i$$
$$+ \sum_{s=1}^{t-1} (-1)^s (\mathbf{s}, \mathbf{t}) \sum P_{k_0,i_0} \cdot \cdots \cdot P_{k_s,i_s} \otimes Q_{i_0} \cdot \cdots \cdot Q_{i_s}$$
$$- \sum_{i_0=m_t+1}^{m_{t+1}} \sum_{i_1=1}^{m_t} P_{t+1,i_0} \cdot P_{r(i_1),i_1} \otimes Q_{i_0} \cdot Q_{i_1}$$
$$- \sum_{i=m_t+1}^{m_{t+1}} \sum_{s=1}^{t-1} (-1)^s (\mathbf{s}, \mathbf{t}) \sum P_{t+1,i} \cdot P_{k_0,i_0} \cdot \cdots \cdot P_{k_s,i_s} \otimes Q_i \cdot Q_{i_0} \cdot \cdots \cdot Q_{i_s}.$$

Here we have used $(\mathbf{s}, \mathbf{t}) \sum$ instead of the multiple sum

$$\sum_{k_0=s+1}^{t} \sum_{k_1=s}^{k_0-1} \cdots \sum_{k_s=1}^{k_{s-1}-1} \sum_{i_0=m_{k_0-1}+1}^{m_{k_0}} \cdots \sum_{i_s=m_{k_s-1}+1}^{m_{k_s}}.$$

Because of

$$- \sum_{i=m_t+1}^{m_{t+1}} \sum_{s=1}^{t-1} (-1)^s (\mathbf{s}, \mathbf{t}) \sum P_{t+1,i} \cdot P_{k_0,i_0} \cdot \cdots \cdot P_{k_s,i_s} \otimes Q_i \cdot Q_{i_0} \cdot \cdots \cdot Q_{i_s}$$
$$= \sum_{s=2}^{t} (-1)^s (\mathbf{s}-\mathbf{1}, \mathbf{t}) \sum \sum_{i=m_t+1}^{m_{t+1}} P_{t+1,i} \cdot P_{k_0,i_0} \cdot \cdots \cdot P_{k_{s-1},i_{s-1}}$$
$$\otimes Q_i \cdot Q_{i_0} \cdot \cdots \cdot Q_{i_{s-1}}$$
$$= \sum_{s=2}^{t} (-1)^s \sum_{k_1=s}^{t} \sum_{k_2=s-1}^{k_1-1} \cdots \sum_{k_s=1}^{k_{s-1}-1} \sum_{i_0=m_t+1}^{m_{t+1}} \sum_{i_1=m_{k_1-1}+1}^{m_{k_1}}$$
$$\cdots \sum_{i_s=m_{k_s-1}+1}^{m_{k_s}} P_{t+1,i_0} \cdot P_{k_1,i_1} \cdot \cdots \cdot P_{k_s,i_s} \otimes Q_{i_0} \cdot \cdots \cdot Q_{i_s}$$
$$= \sum_{s=2}^{t} (-1)^s \sum_{k_0=s+1}^{t+1} \sum_{k_1=s}^{k_0-1} \cdots \sum_{k_s=1}^{k_{s-1}-1} \sum_{i_0=m_{k_0-1}+1}^{m_{k_0}}$$
$$\cdots \sum_{i_s=m_{k_s-1}+1}^{m_{k_s}} P_{k_0,i_0} \cdot \cdots \cdot P_{k_s,i_s} \otimes Q_{i_0} \cdot \cdots \cdot Q_{i_s}$$
$$- \sum_{s=2}^{t-1} (-1)^s \sum_{k_0=s+1}^{t} \sum_{k_1=s}^{k_0-1} \cdots \sum_{k_s=1}^{k_{s-1}-1} \sum_{i_0=m_{k_0-1}+1}^{m_{k_0}}$$
$$\cdots \sum_{j_s=m_{k_s-1}+1}^{m_{k_s}} P_{k_0,i_0} \cdot \cdots \cdot P_{k_s,i_s} \otimes Q_{i_0} \cdot \cdots \cdot Q_{i_s}$$

$$= \sum_{s=2}^{t} (-)^s(\mathbf{s}, \mathbf{t}) \sum P_{k_0, i_0} \cdot \cdots \cdot P_{k_s, i_s} \otimes Q_{i_0} \cdot \cdots \cdot Q_{i_s}$$

$$- \sum_{s=1}^{t-1} (-1)^s(\mathbf{s}, \boldsymbol{t}) \sum P_{k_0, i_0} \cdot \cdots \cdot P_{k_s, i_s} \otimes Q_{i_0} \cdot \cdots \cdot Q_{i_s}$$

$$- (\mathbf{1}, \mathbf{t}) \sum P_{k_0, i_0} \cdot P_{k_1, i_1} \otimes Q_{i_0} \cdot Q_{i_1},$$

the correctness of the Theorem is proven. ∎

3. Bivariate Birkhoff Interpolation

The bivariate Birkhoff interpolation problem (1.1) can be described by a (modified) incidence matrix. Let

$$\mathscr{E}_{m,n} = (E_{i,k})_{1 \leqslant i \leqslant m, 0 \leqslant k \leqslant M}$$

be a $m \times (M+1)$-matrix with the properties

(i) *For exactly* $M+1$ *pairs* $(i, k) \in \{1,..., m\} \times \{0,..., M\}$, *let* $E_{i,k} = (e_{i,j}^{k,l})_{1 \leqslant j \leqslant a_{i,k}, 0 \leqslant l \leqslant N_{i,k}}$ *be* $a_{i,k} \times (N_{i,k}+1)$*-incidence matrices with* $a_{i,k} \in \mathbb{N}$, $N_{i,k} \in \mathbb{N}_0$,

(ii) $E_{i,k} = 0$ *for all other* (i,k). (3.1)

To simplify our notations, we define Z by

$$Z := \{(i, k) \in \{1,..., m\} \times \{0,..., M\} \mid E_{i,k} \neq 0\}. \tag{3.2}$$

For given real numbers $x_i, y_{i,k;j}((i, k) \in Z, \quad 1 \leqslant j \leqslant a_{i,k})$ with $x_1 < \cdots < x_m, y_{i,k;1} < \cdots < y_{i,k;a_{i,k}}$, we consider for sufficiently differentiable $f: \mathbb{R}^2 \to \mathbb{R}$ the functionals

$$D_{i,j}^{k,l} = D_{x_i, y_{i,k;j}}^{k,l}, \qquad (i, k) \in Z, \qquad e_{i,j}^{k,l} = 1. \tag{3.3}$$

Obviously we can identify the matrix $\mathscr{E}_{m,M}$ with the Birkhoff conditions (3.3). Therefore we will call $\mathscr{E}_{m,M}$ with the conditions (3.1) a *two-dimensional incidence matrix*.

There are exactly one $p \in \mathbb{N}$ and a surjective map $g: Z \to \{1,..., p\}$ with

$$g(i_1, k_1) = g(i_2, k_2) \qquad \text{if and only if } N_{i_1,k_1} = N_{i_2,k_2}, \tag{3.4}$$

$$N_{g^{-1}\{1\}} < \cdots < N_{g^{-1}\{p\}}. \tag{3.5}$$

For abbreviation we define for every $s \in \{1,..., p\}$,

$$
\begin{aligned}
N_s &:= N_{g^{-1}\{s\}}, \\
M_s &:= \operatorname{card}\{(i,k) \in Z \mid g(i,k) \leqslant s\} - 1.
\end{aligned}
\tag{3.6}
$$

EXAMPLE 3.1. The matrix

$$
\mathscr{E}_{4,4} = \left[\begin{array}{ccccc}
0 & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & & 0 & 0 \\
\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} & 0 & & 0 & 0 \\
0 & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} & 0 & 0 \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 & 0 & 0 & 0
\end{array}\right]
$$

is a two-dimensional incidence matrix with $Z = \{(1,1), (2,0), (3,1), (3,2), (4,0)\}$ and

$$
\begin{aligned}
&N_{2,0} = N_1 = 4, && M_1 = 0, \\
&N_{1,1} = N_{3,1} = N_2 = 3, && M_2 = 2, \\
&N_{3,2} = N_{4,0} = N_3 = 1, && M_3 = 4.
\end{aligned}
$$

For given $x_1 < \cdots < x_4$ and

$$
\begin{gathered}
y_{2,0;1} < \cdots < y_{2,0;3}, \\
y_{1,1;1} < \cdots < y_{1,1;3}, \\
y_{3,1;1} < \cdots < y_{3,1;4}, \\
y_{3,2;1} < y_{3,2;2}, \\
y_{4,0;1} < y_{4,0;2},
\end{gathered}
$$

$\mathscr{E}_{4,4}$ corresponds to the interpolation conditions $\{D^{k,l}_{x_i, y_{i,k;j}} \mid (i,k) \in Z,\ e^{k,l}_{i,j} = 1\}$.

DEFINITION 3.2. Let $\mathscr{E}_{m,M}$ be a two-dimensional incidence matrix.

(i) We will call a point set

$$\{(x_i, y_{i,k;j}) \mid (i,k) \in Z, 0 \leqslant j \leqslant a_{i,k}\} \subset \mathbb{R}^2 \tag{3.7}$$

a *base set of knots* for $\mathscr{E}_{m,M}$ if the following conditions hold:

$$x_1 < \cdots < x_m,$$

$$y_{i,k;1} < \cdots < x_{i,k;a_{i,k}}, \qquad (i,k) \in Z.$$

(ii) Let K be a base set of knots for $\mathscr{E}_{m,M} \cdot \mathscr{E}_{m,M}$ will be called *conditionally regular with reference* to K if for every set of real numbers $\{\alpha_{i,j}^{k,l} \mid (i,k) \in Z, e_{i,j}^{k,l} = 1\}$ there exists a polynomial $P \in \sum_{s=1}^{p} \Pi_{M_s} \otimes \prod_{N_s}$ (M_s, N_s, $1 \leqslant s \leqslant p$, corresponding to (3.6)) with

$$P^{(k,l)}(x_i, y_{i,k;j}) = \alpha_{i,j}^{k,l}.$$

(iii) $\mathscr{E}_{m,M}$ will be called *regular* if $\mathscr{E}_{m,M}$ is conditionally regular with reference to every base set of knots.

Now we will deduce sufficient conditions for regularity of two-dimensional incidence matrices by the method used in Section 1.

THEOREM 3.3. *Let $\mathscr{E}_{m,M}$ be a two-dimensional $m \times (M+1)$-incidence matrix as well as $(x_1, ..., x_m) \in \mathbb{R}^m$ a knot sequence (i.e., $x_1 < \cdots < x_m$). For every $s \in \{1, ..., p\}$ let the (one-dimensional) incidence matrices*

$$E_s = (e_{i,k})_{1 \leqslant i \leqslant m, 0 \leqslant k \leqslant m_s} \text{ with } e_{i,k} = 1 \text{ if and only if } (i,k) \in Z \text{ and } N_{i,k} \geqslant N_s, \tag{3.8}$$

be conditionally regular with reference to $(x_1, ..., x_m)$. Furthermore, for every $(i,k) \in Z$ let the $a_{i,k} \times (N_{i,k}+1)$-incidence matrices $E_{i,k}$ be conditionally regular with reference to a knot sequence $(y_{i,k;1}, ..., y_{i,k;a_{i,k}})$. Then $\mathscr{E}_{m,M}$ is regular with reference to the base set of knots

$$\{(x_i, y_{i,k;j}) \mid (i,k) \in Z, 1 \leqslant j \leqslant a_{i,k}\}.$$

Proof. We have to show that the interpolation problem

$$\left(\mathscr{C}^q(\mathbb{R}^2), \sum_{s=1}^{p} \prod_{M_s} \otimes \prod_{N_s}; D_{x_i, y_{i,k;j}}^{k,l}: (i,k) \in Z, e_{i,j}^{k,l} = 1\right)$$

with $q=\max_{1\leqslant s\leqslant p}\{M_s+N_s\}$ is uniquely solvable. According to the assumptions, the interpolation problems

$$(\mathscr{C}^{M_s}(\mathbb{R}), \textstyle\prod_{M_s};\, D^k_{x_i}\colon N_{i,k}\geqslant N_s), \qquad 1\leqslant s\leqslant p,$$

and

$$(\mathscr{C}^{N_s}(\mathbb{R}), \textstyle\prod_{N_s};\, D^l_{y_{i,k;j}}\colon e^{k,l}_{i,j}=1), \qquad (i,k)\in Z,$$

are uniquely solvable. Because of $D^{k,l}_{x_i,y_{i,k;j}}=D^k_{x_i}\otimes D^l_{y_{i,k;j}}$, Theorem 3.3 follows from Theorem 2.1. ▌

COROLLARY 3.4. *If the incidence matrices E_s of* (3.8) *are regular for all $1\leqslant s\leqslant p$ and also the incidence matrices $E_{i,k}$, $(i,k)\in Z$, then $\mathscr{E}_{m,M}$ is regular.*

EXAMPLE 3.5. We will investigate the incidence matrix $\mathscr{E}_{4,4}$ in Example 3.1 for regularity: The incidence matrices $E_{i,k}$, $(i,k)\in Z$, are all regular. Also the incidence matrices

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \begin{pmatrix}0&1&0\\1&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&1&1&0&0\\1&0&0&0&0\end{pmatrix}$$

are regular. Therefore $\mathscr{E}_{4,4}$ is regular and the interpolation problem

$$(\mathscr{C}^5(\mathbb{R}^2), \textstyle\prod_0\otimes\prod_4+\prod_2\otimes\prod_3+\prod_4\otimes\prod_1;\, D^{k,l}_{x_i,y_{i,k;j}}\colon (i,k)\in Z,$$
$$e^{k,l}_{i,j}=1, \qquad x_1<\cdots<x_4,\, y_{i,k;1}<\cdots<y_{i,k;a_{i,k}})$$

is uniquely solvable.

4. REMARKS

(i) Let the two-dimensional $M\times(N+1)$-incidence matrix $\mathscr{E}_{M,N}$ be equal to $(E_{i,k})_{1\leqslant i\leqslant M,0\leqslant k\leqslant N}$ with the regular $m_{i,k}\times(n_{i,k}+1)$-incidence $E_{i,k}=(e^{k,l}_{i,j})_{1\leqslant j\leqslant m_{i,k},\,0\leqslant l\leqslant n_{i,k}}$, $1\leqslant i\leqslant M$, $0\leqslant k\leqslant N_i$ $(N_i\in\mathbb{N}_0$, $\sum_{i=1}^M(N_i+1)=N+1)$, $m_{i,k}\in\mathbb{N}$, $n_{i,k}\in\mathbb{N}_0$, and $E_{i,k}=0$, for $k>N_i$. If

$$n_{i,k_1}\geqslant n_{i,k_2} \tag{4.1}$$

holds for $k_1<k_2\leqslant N_i$, $1\leqslant i\leqslant M$, then $\mathscr{E}_{M,N}$ is regular.

Proof. There exist $p, n_1, ..., n_p \in \mathbb{N}$, $n_1 < \cdots < n_p$, so that $n_{i,j} \in \{n_1, ..., n_p\}$ for all $1 \leqslant i \leqslant M$, $0 \leqslant j \leqslant N_i$. For $i \in \{1, ..., p\}$, we define

$$Z_i := \{(j, l) \mid n_{j,l} \geqslant n_i,\ 1 \leqslant j \leqslant M,\ 0 \leqslant 1 \leqslant N_i\},$$

$$m_i := \mathrm{card}(Z_i) - 1.$$

Because of (4.1) the interpolation problems

$$(\mathscr{C}^M(\mathbb{R}), \textstyle\prod_{m_i}; D^l_{x_j} : (j, l) \in Z_i, x_1, ..., x_m \in \mathbb{R}, x_i \neq x_j \text{ for } i \neq j)$$

are Hermite interpolation problems and therefore uniquely solvable. With Theorem 2.1 the unique solvability of the interpolation problem

$$(\mathscr{C}^q(\mathbb{R}^2), \sum_{s=1}^{p} \textstyle\prod_{m_s} \otimes \prod_{n_s}; D^{k,l}_{x_i, y_{i,k;j}} : 1 \leqslant i \leqslant M, 0 \leqslant k \leqslant N_i, e^{k,l}_{i,j} = 1,$$

$$x_1, ..., x_m \in \mathbb{R}, x_i \neq x_j \text{ for } i \neq j, y_{i,k;1} < \cdots < y_{i,k;m_{i,k}})$$

$(q = \max_{1 \leqslant s \leqslant p}\{m_s + n_s\})$ is shown. ∎

(ii) In [13] Haussmann and Knoop have shown that the following two-dimensional incidence matrix

$$\mathscr{E} = \begin{pmatrix} E_1 & \cdots & E_1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ E_M & \cdots & E_M & 0 & \cdots & 0 \end{pmatrix},$$

which always has the same Hermite matrices $E_i = (e_{k,l})_{1 \leqslant k \leqslant m_i, 0 \leqslant 1 \leqslant n_i, k}$ in the first N_i columns of the ith row, is regular with reference to every set of knots

$$\{(x_i, y_{i,j}) \mid 1 \leqslant i \leqslant M, 1 \leqslant j \leqslant m_i\}.$$

From Remark (i), it follows directly that $\mathscr{E}$ is regular with reference to every base set

$$\{(x_i, y_{i,k;j}) \mid 1 \leqslant i \leqslant M, 0 \leqslant k \leqslant N_i, 1 \leqslant j \leqslant m_i\}.$$

(iii) The interpolation method given in Section 2 covers all those results of multivariate interpolation deduced with Boolean methods given in [1–3], and in [5–11]. On the other hand, there exists a simple example solvable with this new method but that cannot be investigated by Boolean methods: We apply the interpolation conditions

$$\{D^{k,l}_z \mid z \in \{0, \tfrac{1}{2}, 1\}^2 \setminus \{(\tfrac{1}{2}, \tfrac{1}{2})\}, 0 \leqslant k + 1 \leqslant 1\}. \tag{4.2}$$

The incidence matrix corresponding to (4.2) is the following:

$$\left[\begin{array}{cccc} \begin{pmatrix} 1&1&0&0&0&0\\ 1&1&0&0&0&0\\ 1&1&0&0&0&0 \end{pmatrix} & \begin{pmatrix} 1&0&0\\ 1&0&0\\ 1&0&0 \end{pmatrix} & 0\quad 0\quad 0\quad 0 \\ \begin{pmatrix} 1&1&0&0\\ 1&1&0&0 \end{pmatrix} & \begin{pmatrix} 1&0\\ 1&0 \end{pmatrix} & 0\quad 0\quad 0\quad 0 \\ \begin{pmatrix} 1&1&0&0&0&0\\ 1&1&0&0&0&0\\ 1&1&0&0&0&0 \end{pmatrix} & \begin{pmatrix} 1&0&0\\ 1&0&0\\ 1&0&0 \end{pmatrix} & 0\quad 0\quad 0\quad 0 \end{array}\right]$$

With Remark (i) the regularity of the matrix is shown and the interpolation space is $H = \Pi_1 \otimes \prod_5 + \prod_2 \otimes \prod_3 + \prod_4 \otimes \prod_2 + \prod_5 \otimes \prod_1$. Especially the interpolation problem

$$(\mathscr{C}^1(\mathbb{R}^2), H; D_z^{k,l}: z \in \{0, \tfrac{1}{2}, 1\}^2 \setminus \{(\tfrac{1}{2}, \tfrac{1}{2})\}, 0 \leqslant k + 1 \leqslant 1\}$$

is uniquely solvable.

(iv) The bivariate Birkhoff interpolation can be generalized in a canonical way. If we assume in (3.1) (i) that for $(i, k) \in Z$ the elements $E_{i,k}$ are n-dimensional incidence matrices ($n \in \mathbb{N}$), we get recursively a $(n+1)$-dimensional incidence matrix.

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